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A MATHEMATICAL DESCRIPTION OF MACROSCOPIC BEHAVIOUR OF BRICK MASONRY

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Abstract—A mathematical formulation is presented for the description of the average mechanical properties of structural masonry. The conceptual approach is based on the framework recently outlined by Pietruszczak (1991). An element of structural masonry is regarded as a composite medium consisting of the brick matrix intercepted by the sets of head and bed joints. The former are considered as aligned, uniformly dispersed weak inclusions, whereas the latter represent continuous planes of weakness. A general three-dimensional formulation is provided and is subsequently applied to estimate the average macroscopic properties in the elastic range and to investigate the conditions at failure. An extensive numerical study is performed.

1. INTRODUCTION

Over the last few decades the research in structural masonry has concentrated mainly on the experimental testing of brickworks. The results of those investigations have provided valuable information used to establish empirically or semi-empirically based methodologies for the design of masonry structures. There have been only a few isolated attempts to estimate the properties of masonry in a rigorous analytical manner [e.g. Pande *et al.* (1989)]. It is quite apparent however, that an adequate description of these properties is essential for the analysis of complex boundary value problems involving masonry structures.

The mechanical response of masonry can be analyzed by employing the finite element technique. By using the physical and the actual geometric properties of brick units and mortar, the numerical solution to a class of selected problems can be obtained (Ali and Page, 1989; Afshari and Kaldjian, 1989). There are however serious limitations to this approach. Firstly, the actual geometry of the brickwork may result in ill-conditioning of the algebraic system and/or instability of the numerical solution. Secondly, the approach becomes quite impractical in the context of large-scale masonry structures comprising a very large number of brick units subjected to a three-dimensional state of stress.

This paper presents an alternative approach for the description of the behaviour of structural masonry. The methodology followed is based on the framework outlined by Pietruszczak (1991). A typical element of brickwork is regarded as a structured/composite medium for which the average macroscopic properties can be uniquely identified. Thus, a representative volume of the "material" considered is assumed to consist of a number of brick units intercepted by two orthogonal families of joints. The presence of discrete sets of mortar joints results in a strong directional dependence of the average mechanical properties. The estimate of these properties is the main interest here. The paper is written in the following sequence. First, a general three-dimensional formulation is provided. The average constitutive relation is derived by employing the assumption that the head joints represent a set of aligned weak inclusions and the bed joints form continuous planes of weakness. The formulation is then applied to establish the average elastic properties of the system. Later, the phenomenon of a progressive failure of the brickwork is investigated. An extensive numerical study is carried out; the performance of the framework is verified for a series of biaxial compression–tension and compression–compression tests.

2. MACROSCOPIC RESPONSE OF STRUCTURAL MASONRY; MATHEMATICAL FORMULATION

Consider a typical element of structural masonry, i.e. a brick panel, as shown schematically in Fig. 1a, subjected to a uniformly distributed load. On the macroscale, the

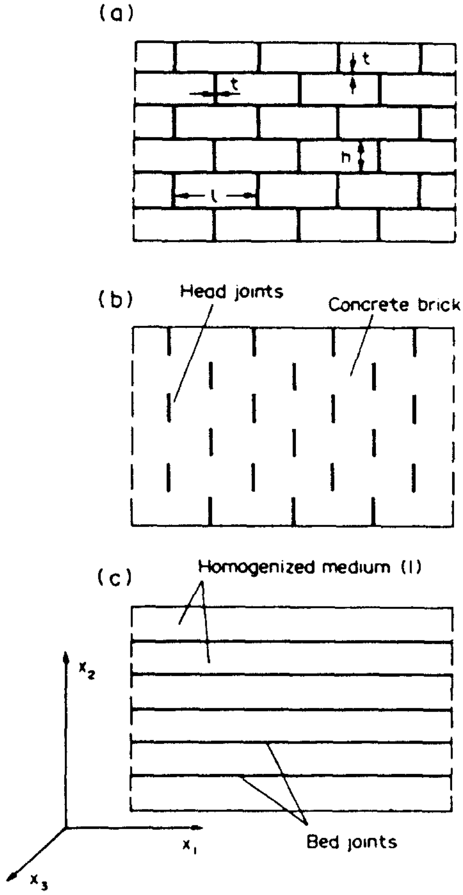

Fig. 1. (a) Geometry of a structural masonry panel; (b) Medium (1); (c) Medium (1) intercepted by bed joints (after Pietruszczak, 1991).

panel can be regarded as a two-phase composite consisting of brick units interspersed by two orthogonal sets of joints filled with mortar. In order to describe the average mechanical properties of the system, it is convenient to address the influence of head (vertical) and bed joints separately, i.e. invoke the concept of a superimposed medium.

Referring to Fig. 1b, consider first the brick matrix with a family of head joints (a so-called medium (1)). The head joints can be treated as aligned, uniformly dispersed weak inclusions embodied in the matrix. The average properties of the medium (1) can be represented by a constitutive relation

$$\dot{\sigma}^{(1)} = [D^{(1)}]\dot{\varepsilon}^{(1)} \tag{1}$$

where $\dot{\sigma}^{(1)} = \{\dot{\sigma}_{11}^{(1)}, \dot{\sigma}_{22}^{(1)}, \dot{\sigma}_{33}^{(1)}, \dot{\sigma}_{12}^{(1)}, \dot{\sigma}_{13}^{(1)}, \dot{\sigma}_{23}^{(1)}\}^{\mathrm{T}}$ and $\dot{\varepsilon}^{(1)} = \{\dot{\varepsilon}_{11}^{(1)}, \dot{\varepsilon}_{22}^{(1)}, \dot{\varepsilon}_{33}^{(1)}, \dot{\gamma}_{12}^{(1)}, \dot{\gamma}_{13}^{(1)}, \dot{\gamma}_{23}^{(1)}\}^{\mathrm{T}}$ are the volume averages of stress/strain rates in (1). In particular, the homogenized medium (1) can be regarded as an orthotropic elastic–brittle material. In such a case, the components of $[D^{(1)}]$ can be estimated from Eshelby's (1957) solution to an ellipsoidal inclusion problem combined with Mori-Tanaka's (1973) mean-field theory. The details concerning the specification of $[D^{(1)}]$ matrix and the criterion for an elastic–brittle transition are discussed later in this paper.

The entire masonry panel can now be represented by a homogenized medium (1) stratified by a family of bed joints (2), Fig. 1c. The bed joints run continuously through the panel and form the weakest link in the microstructure of the system. In particular, the bed joints can be regarded as an elastoplastic medium with mechanical properties defined by

$$\dot{\sigma}^{(2)} = [D^{(2)}]\dot{\varepsilon}^{(2)}. \tag{2}$$

Assuming that both constituents (1) and (2) exist simultaneously and are perfectly bonded, the overall stress/strain rate averages $\dot{\sigma}$ and $\dot{\varepsilon}$ can be derived from the averaging rule (Hill, 1963)

$$\dot{\varepsilon} = \eta_1\dot{\varepsilon}^{(1)} + \eta_2\dot{\varepsilon}^{(2)} \tag{3}$$

$$\dot{\sigma} = \eta_1\dot{\sigma}^{(1)} + \eta_2\dot{\sigma}^{(2)} \tag{4}$$

where ηs are the volume fractions of both constituents,

$$\eta_1 = \frac{h}{h+t}; \quad \eta_2 = \frac{t}{h+t} \tag{5}$$

and h and t represent the spacing and the thickness of bed joints, respectively.

The assumption of perfect bonding between the constituents and the equilibrium requirements provides additional kinematic and static constraints

$$[\delta^*]\dot{\varepsilon}^{(1)} = [\delta^*]\dot{\varepsilon}^{(2)} \tag{6}$$

$$[\delta]\dot{\sigma}^{(1)} = [\delta]\dot{\sigma}^{(2)} \tag{7}$$

where

$$[\delta^*] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad [\delta] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

The constraints (6) and (7), as applied to averages, are rigorous provided $t \ll h$. Their validity can easily be verified from the Eshelby's equivalence principle.

It is evident that the field equations listed above (eqns (1)–(4), together with (6) and (7)) provide a set of 30 equations for 30 unknowns, e.g. $\dot{\sigma}$, $\dot{\sigma}^{(1)}$, $\dot{\sigma}^{(2)}$, $\dot{\varepsilon}^{(1)}$ and $\dot{\varepsilon}^{(2)}$. Thus, the problem is mathematically determinate. It should be noted that the total number of unknowns can be reduced by introducing certain simplifying assumptions pertaining to the kinematics of bed joints. The formulation discussed by Pietruszczak (1991) for example, has been derived by expressing the local deformation field in bed joints in terms of velocity discontinuities rather than strain rates $\dot{\varepsilon}^{(2)}$, thereby reducing the number of unknowns to 27.

In order to solve the problem, i.e. provide an explicit form of the average constitutive relation, it is convenient to introduce the following identity

$$[\delta]\dot{\sigma}^{(i)} = [\delta][D^{(i)}]\dot{\varepsilon}^{(i)} = [E^{(i)}][\delta^*]\dot{\varepsilon}^{(i)} + [F^{(i)}][\delta]\dot{\varepsilon}^{(i)}; \quad i = 1,2 \tag{9}$$

where

$$[E^{(i)}] = \begin{bmatrix} D^{(i)}_{21} & D^{(i)}_{23} & D^{(i)}_{25} \\ D^{(i)}_{41} & D^{(i)}_{43} & D^{(i)}_{45} \\ D^{(i)}_{61} & D^{(i)}_{63} & D^{(i)}_{65} \end{bmatrix}; \quad [F^{(i)}] = \begin{bmatrix} D^{(i)}_{22} & D^{(i)}_{24} & D^{(i)}_{26} \\ D^{(i)}_{42} & D^{(i)}_{44} & D^{(i)}_{46} \\ D^{(i)}_{62} & D^{(i)}_{64} & D^{(i)}_{66} \end{bmatrix}. \tag{10}$$

Utilizing eqns (9) and (6), the static constraint (7) can now be expressed in the form

$$[E^{(1)}][\delta^*]\dot{\varepsilon} + [F^{(1)}][\delta]\dot{\varepsilon}^{(1)} = [E^{(2)}][\delta^*]\dot{\varepsilon} + [F^{(2)}][\delta]\dot{\varepsilon}^{(2)}. \tag{11}$$

Given the representation (11) and the decomposition (3), the strain rates in both constituents can be uniquely related to $\dot{\varepsilon}$. In view of kinematic constraints (6), the set of equations (3) reduces to

$$[\delta]\dot{\varepsilon}^{(2)} = \frac{1}{\eta_2}[\delta]\dot{\varepsilon} - \frac{\eta_1}{\eta_2}[\delta]\dot{\varepsilon}^{(1)}. \tag{12}$$

Substitution of eqn (12) in eqn (11) results, after some simple algebra, in

$$[\delta]\dot{\varepsilon}^{(1)} = [\bar{S}]\dot{\varepsilon}; \tag{13}$$

where

$$[\bar{S}] = \left([F^{(1)}] + \frac{\eta_1}{\eta_2}[F^{(2)}]\right)^{-1} \left\{\frac{1}{\eta_2}[F^{(2)}][\delta] + ([E^{(2)}] - [E^{(1)}])[\delta^*]\right\}. \tag{14}$$

Thus, in view of eqn (3), the following relationship is obtained

$$\dot{\varepsilon}^{(1)} = [S_1]\dot{\varepsilon} \tag{15}$$

where

$$[S_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_{15} & \bar{S}_{16} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \bar{S}_{31} & \bar{S}_{32} & \bar{S}_{33} & \bar{S}_{34} & \bar{S}_{35} & \bar{S}_{36} \end{bmatrix} \tag{16}$$

and the components of $[\bar{S}]$ are defined by eqn (14).

The strain rates in bed joints can be expressed in a similar functional form to that of eqn (15). After substituting eqn (15) in eqn (3) and solving for $\dot{\varepsilon}^{(2)}$, one obtains

$$\dot{\varepsilon}^{(2)} = [S_2]\dot{\varepsilon} \tag{17}$$

where

$$[S_2] = \left(\frac{1}{\eta_2}[I] - \frac{\eta_1}{\eta_2}[S_1]\right) \tag{18}$$

and $[I]$ represents the unit matrix (6×6).

Finally, the overall stress rate averages $\dot{\sigma}$ can be derived from eqn (4). Substitution of eqns (15) and (17) in eqn (4), results in

$$\dot{\sigma} = \{\eta_1[D^{(1)}][S_1] + [D^{(2)}]([I] - \eta_1[S_1])\}\dot{\varepsilon} = [D]\dot{\varepsilon}. \tag{19}$$

The above equation represents the average constitutive relation for the entire composite system. As expected, the macroscopic behaviour depends on the mechanical properties of both constituents and their volume contributions. In the following sections the proposed mathematical framework is investigated in detail. First, the average elastic properties of the masonry are established and subsequently the phenomenon of progressive failure of the material microstructure is addressed.

3. AVERAGE ELASTIC PROPERTIES OF STRUCTURAL MASONRY

Assume that all constituents in the microstructure remain elastic and determine the average elastic properties of the composite. Consider first the medium (1), i.e. brick matrix with uniformly dispersed head joints in the form of monotonically aligned rectangular parallelepipeds. If both the bricks and the joints are isotropic then the medium (1), as a whole, will become orthotropic. In this case, the constitutive matrix, eqn (1), assumes the form

$$[D^{(1)}] = \begin{bmatrix} D_{11}^{(1)} & D_{12}^{(1)} & D_{13}^{(1)} & 0 & 0 & 0 \\ D_{12}^{(1)} & D_{22}^{(1)} & D_{23}^{(1)} & 0 & 0 & 0 \\ D_{13}^{(1)} & D_{23}^{(1)} & D_{33}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66}^{(1)} \end{bmatrix}. \tag{20}$$

The nine independent elastic constants are functions of the properties of both constituents as well as the cross-sectional aspect ratio and the volume fraction of the inclusions. Recently, Zhao and Weng (1990) have identified the average elastic constants of an orthotropic composite reinforced with aligned elliptic cylinders. The estimates are based on Eshelby's solution to the ellipsoidal inclusion problem combined with Mori-Tanaka's mean-field theory (to deal with the finite concentration of inclusions). The results reported by Zhao and Weng can be applied to estimate the average elastic properties of medium (1), viz. eqn (20). The algebraic expressions defining the elastic constants are quite complex and will not be cited here. The reader is referred, in this respect, directly to the original publication.

Assume now that the bed joints, eqn (2), are considered as isotropic, i.e.

$$[D^{(2)}] = \begin{bmatrix} D_{11}^{(2)} & D_{12}^{(2)} & D_{12}^{(2)} & 0 & 0 & 0 \\ D_{12}^{(2)} & D_{11}^{(2)} & D_{12}^{(2)} & 0 & 0 & 0 \\ D_{12}^{(2)} & D_{12}^{(2)} & D_{11}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44}^{(2)} \end{bmatrix}; \quad D_{44}^{(2)} = D_{11}^{(2)} - D_{12}^{(2)}. \tag{21}$$

Given both representations (2) and (21) the matrices $[E^{(i)}]$ and $[F^{(i)}]$, defined in eqn (10), reduce to

$$[E^{(1)}] = \begin{bmatrix} D_{12}^{(1)} & D_{23}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad [F^{(1)}] = \begin{bmatrix} D_{22}^{(1)} & 0 & 0 \\ 0 & D_{44}^{(1)} & 0 \\ 0 & 0 & D_{66}^{(1)} \end{bmatrix} \tag{22}$$

$$[E^{(2)}] = \begin{bmatrix} D_{12}^{(2)} & D_{12}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad [F^{(2)}] = \begin{bmatrix} D_{11}^{(2)} & 0 & 0 \\ 0 & D_{44}^{(2)} & 0 \\ 0 & 0 & D_{44}^{(2)} \end{bmatrix}. \tag{23}$$

Substituting the above representations in eqn (14), after simple algebraic manipulations one obtains

$$[\bar{S}] = \begin{bmatrix} (D_{12}^{(2)} - D_{12}^{(1)})/a; & \frac{1}{\eta_2} D_{11}^{(2)}/a; & (D_{12}^{(2)} - D_{23}^{(1)})/a; & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\eta_2} D_{44}^{(2)}/b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\eta_2} D_{44}^{(2)}/c \end{bmatrix} \tag{24}$$

where

$$a = D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}; \quad b = D_{44}^{(1)} + \frac{\eta_1}{\eta_2} D_{44}^{(2)}; \quad c = D_{66}^{(1)} + \frac{\eta_1}{\eta_2} D_{44}^{(2)}.$$

Thus, given the definitions (20), (21) and (24), the components of the macroscopic constitutive matrix can be determined from eqn (19). The composite panel is an orthotropic body (on a macroscale) and the nine independent components of $[D]$ matrix are defined as

$$D_{11} = (\eta_1 D_{11}^{(1)} + \eta_2 D_{11}^{(2)}) - \frac{\eta_1 (D_{12}^{(2)} - D_{12}^{(1)})^2}{D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}}; \quad D_{22} = \frac{1}{\frac{1}{\eta_1} D_{22}^{(1)} + \frac{1}{\eta_2} D_{11}^{(2)}}$$

$$D_{33} = (\eta_1 D_{33}^{(1)} + \eta_2 D_{11}^{(2)}) - \frac{\eta_1 (D_{12}^{(2)} - D_{23}^{(1)})^2}{D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}}; \quad D_{44} = \frac{1}{\frac{1}{\eta_1} D_{44}^{(1)} + \frac{1}{\eta_2} (D_{11}^{(2)} - D_{12}^{(2)})}$$

$$D_{55} = \eta_1 D_{55}^{(1)} + \eta_2 (D_{11}^{(2)} - D_{12}^{(2)}); \quad D_{66} = \frac{1}{\frac{1}{\eta_1} D_{66}^{(1)} + \frac{1}{\eta_2} (D_{11}^{(2)} - D_{12}^{(2)})}$$

$$D_{12} = (\eta_1 D_{12}^{(1)} + \eta_2 D_{12}^{(2)}) - \frac{\eta_1 (D_{12}^{(2)} - D_{12}^{(1)})(D_{11}^{(2)} - D_{22}^{(1)})}{D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}}$$

$$D_{13} = (\eta_1 D_{13}^{(1)} + \eta_2 D_{12}^{(2)}) - \frac{\eta_1 (D_{12}^{(2)} - D_{12}^{(1)})(D_{12}^{(2)} - D_{23}^{(1)})}{D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}}$$

$$D_{23} = (\eta_1 D_{23}^{(1)} + \eta_2 D_{12}^{(2)}) - \frac{\eta_1 (D_{12}^{(2)} - D_{23}^{(1)})(D_{11}^{(2)} - D_{22}^{(1)})}{D_{22}^{(1)} + \frac{\eta_1}{\eta_2} D_{11}^{(2)}}. \tag{25}$$

Numerical examples

In order to illustrate the mathematical framework outlined in this section, some numerical simulations were carried out. The objective was to investigate the influence of the joint thickness and the elastic properties of the constituents (brick and mortar) on the average elastic response of the masonry panel. Both constituents were assumed to be isotropic and the elastic constants (Young's modulus, E and Poisson's ratio, ν) were selected after Pande *et al.* (1989) as,

$$E_b = 1.1 \times 10^4 \text{ N/mm}^2; \quad \nu_b = 0.25; \quad \nu_m = 0.20$$

where the subscripts b and m refer to brick and mortar, respectively. The brick dimensions were taken as: $h = 75$ mm, $l = 225$ mm.

Figure 2 shows the variation of nine elastic constants (normalized with respect to properties of the brick) of the masonry panel as a function of the thickness, t, of the joints. The simulations were carried out for different E_b/E_m ratios ranging from 1.1 to 11. It is evident from the figure that an increase in the joint thickness results in a progressive reduction of average elastic moduli (E and G), whereas an increase in the Young's modulus of the mortar causes a corresponding increase in these values.

The contribution of the head joints to the average macroscopic properties of the masonry panel is investigated further in Figs 3 and 4. The results shown in Fig. 3 correspond to the case in which the head joints are treated as continuous vertical planes of weakness ($\alpha \to 0$, where α is the cross-sectional aspect ratio of the head joints, Zhao and Weng, 1990). In other words, the masonry panel is regarded as an elastic medium intercepted by two mutually orthogonal families of continuous joints. The latter approximation was employed by Pande *et al.* (1989) to estimate the average elastic properties of masonry through a direct strain energy approach. By comparing the results with those in Fig. 2, it is evident that the predictions are very close and only the values of E_{11} and G_{31} are underestimated. Thus, the treatment of head joints as weakness planes provides a reasonable approximation in the context of the elastic response of the system.

Finally, Fig. 4 shows the most conservative prediction corresponding to the case when

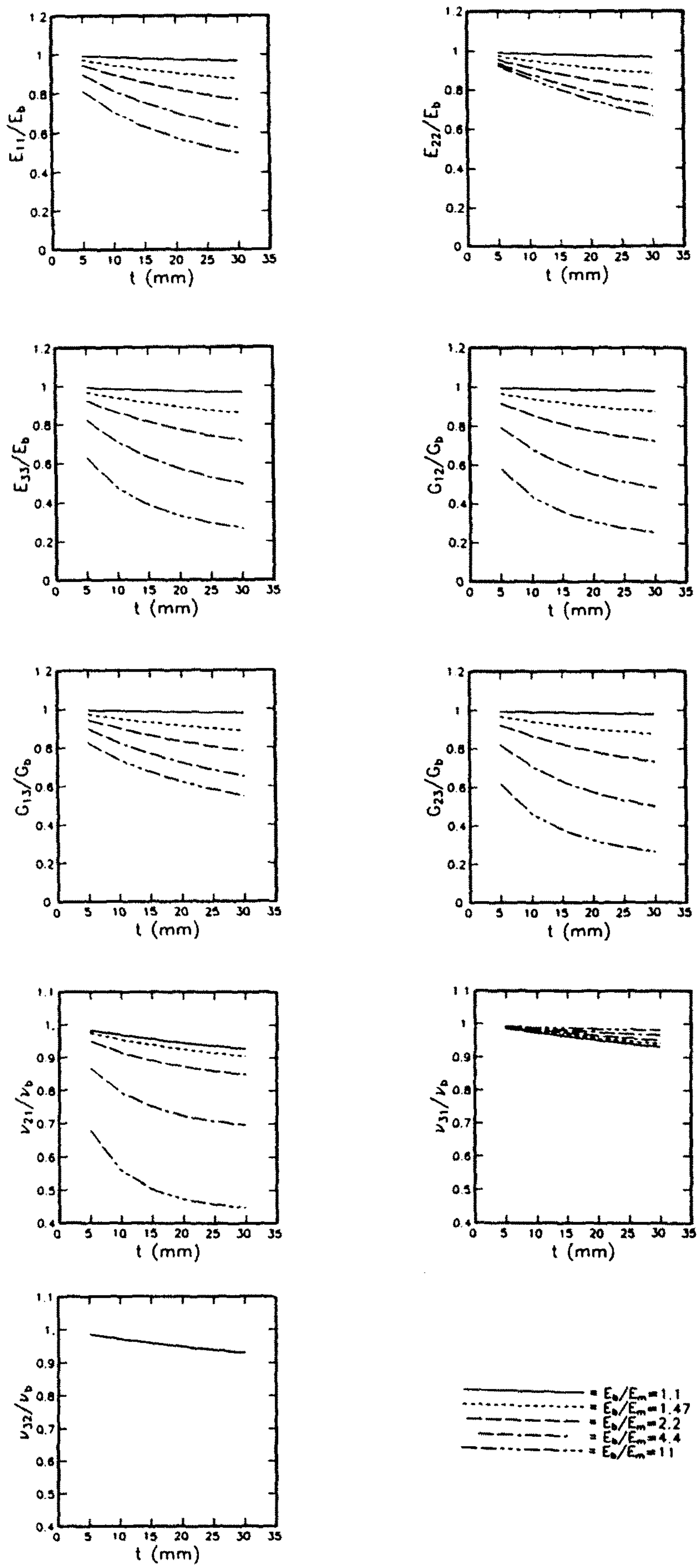

Fig. 2. Average elastic properties of structural masonry.

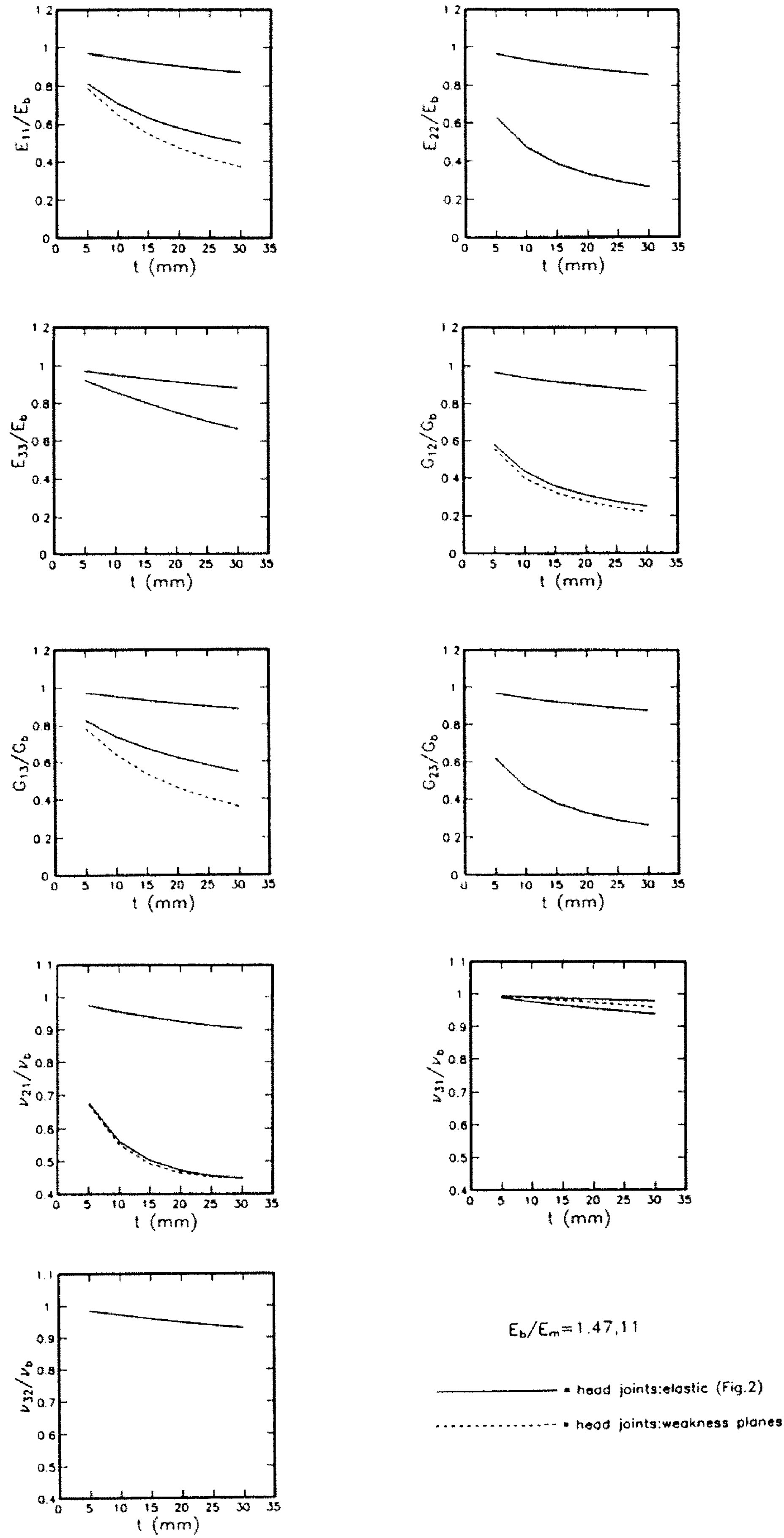

Fig. 3. Influence of the head joints on the average elastic properties of masonry.

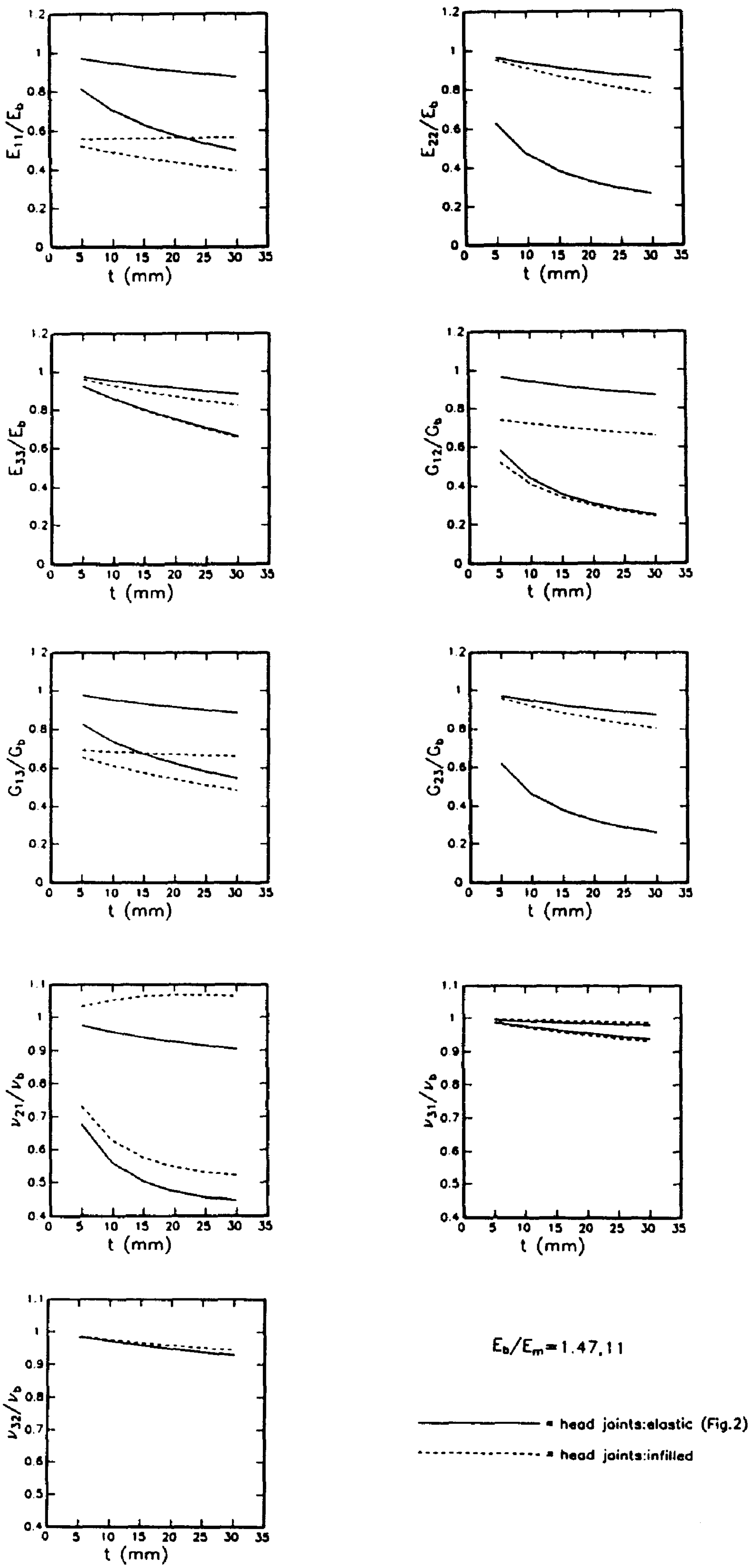

Fig. 4. Influence of the head joints on the average elastic properties of masonry.

the head joints are infilled, i.e. are regarded as voids ($E_m \to 0$). The values of the overall moduli (E and G) are, in general, reduced as compared to those in Fig. 2, in particular E_{11} and G_{31} are affected. It is clear however, that the masonry structure with head joints infilled is still capable of resisting the external load.

4. DESCRIPTION OF PROGRESSIVE FAILURE OF STRUCTURAL MASONRY

The collapse of a masonry panel can result either from the failure of the brick matrix, which is usually of a brittle nature, or from the ductile/brittle failure of the bed joints. The head joints represent a less significant link in the panel microstructure, in the sense that their local failure will not induce the collapse on a macroscale. Thus, it seems reasonable to regard the medium (1) as an orthotropic elastic body, eqn (20), and impose an appropriate criterion for the elastic-brittle transition in the bricks. At the same time, the bed joints can be treated as an elastoplastic strain-hardening material.

Consider first the homogenized medium (1). In order to determine the stress rates in the bricks, express the averaging procedure, eqns (3) and (4), as

$$\dot{\sigma}^{(1)} = \eta'\dot{\sigma}' + \eta''\dot{\sigma}'' \tag{26}$$

$$\dot{\varepsilon}^{(1)} = \eta'\dot{\varepsilon}' + \eta''\dot{\varepsilon}''. \tag{27}$$

Here, the prime and double-prime superscripts refer to brick matrix and mortar (head) joints respectively, whereas ηs are the volume fractions of both constituents

$$\eta' = \frac{l}{l+t}; \quad \eta'' = \frac{t}{l+t} \tag{28}$$

where l represents the spacing of the head joints.

With all the constituents remaining elastic, i.e.

$$\dot{\sigma}' = [D']\dot{\varepsilon}'; \quad \dot{\sigma}'' = [D'']\dot{\varepsilon}''; \quad \dot{\sigma}^{(1)} = [D^{(1)}]\dot{\varepsilon}^{(1)} \tag{29}$$

the stress decomposition, eqn (26), yields

$$[D^{(1)}]\dot{\varepsilon}^{(1)} = \eta'[D']\dot{\varepsilon}' + \eta''[D'']\dot{\varepsilon}''. \tag{30}$$

Thus, substituting eqn (27) in eqn (30) and rearranging

$$\dot{\varepsilon}' = [S']\dot{\varepsilon}^{(1)}; \quad [S'] = \frac{1}{\eta'}([D'] - [D''])^{-1}([D^{(1)}] - [D'']) \tag{31}$$

so that the stress rates in the brick matrix are defined as

$$\dot{\sigma}' = [D']\dot{\varepsilon}' = [D'][S']\dot{\varepsilon}^{(1)}. \tag{32}$$

The failure criterion for the bricks can be expressed in terms of a path-independent condition

$$F(\sigma') = 0 \tag{33}$$

in which $F = 0$ is a scalar-valued function of the basic invariants of σ'. In particular, the functional form proposed by Pietruszczak *et al.* (1988) may be used

$$F = a_1\left(\frac{\sqrt{J_2}}{g(\theta)f_c}\right) + a_2\left(\frac{\sqrt{J_2}}{g(\theta)f_c}\right)^2 - \left(a_3 - \frac{I}{f_c}\right) = 0 \tag{34}$$

where I is the first stress invariant, J_2 is the second invariant of the stress deviator and θ is the angle measure of the third deviatoric stress invariant J_3,

$$\theta = \tfrac{1}{3}\sin^{-1}\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right); \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}. \tag{35}$$

The parameters a_1 through a_3 are dimensionless material constants, whereas f_c represents uniaxial compressive strength of the brick. The function $g(\theta)$, eqn (34), can be selected in the form

$$g(\theta) = \frac{(\sqrt{(1+a)}-\sqrt{(1-a)})K}{K\sqrt{(1+a)}-\sqrt{(1-a)}+(1-K)\sqrt{(1-a\sin 3\theta)}}; \quad K = 1-K_0\,e^{-K_1(a_3-I/f_c)} \tag{36}$$

which satisfies $g(\pi/6) = 1$, $g(-(\pi/6)) = K$ and for $a = 0.999$ guarantees convexity for $K > 0.56$.

Consider now the response of the bed joints. Assuming that joints are elastoplastic, the components of $[D^{(2)}]$, eqn (2), can be derived from the standard plasticity formalism based on the existence of the yield and plastic potential functions

$$f = f(\sigma^{(2)}, \kappa) = 0; \quad \psi = \psi(\sigma^{(2)}) = \text{const.} \tag{37}$$

where κ is a scalar parameter recording the history of plastic deformation, i.e. $\kappa = \kappa\{(\varepsilon^{(2)})^p\}$. In particular, the properties of the mortar can be described by one of the existing formulations applicable to brittle-plastic materials [see e.g. Chen and Han (1988) and Pietruszczak *et al.* (1988)].

By inspecting the geometry of typical structural panels, it is evident that the thickness of the bed joints is small compared with other dimensions. In such a case, the analysis may be simplified by assuming both expressions (37) in the functional form

$$f = \sqrt{(\sigma_{12}^{(2)})^2+(\sigma_{23}^{(2)})^2} - \mu(\sigma_{22}^{(2)}+c) = 0$$
$$\psi = \sqrt{(\sigma_{12}^{(2)})^2+(\sigma_{23}^{(2)})^2} - \bar{\mu}\sigma_{22}^{(2)} = \text{const.} \tag{38}$$

which is analogous to Coulomb friction law. In eqn (38), $\bar{\mu}$ is a constant whereas $\mu = \mu(\xi)$, where ξ is a suitably chosen hardening parameter. In particular, one can select

$$\mu = \mu_0 + (\mu_f - \mu_0)\frac{\xi}{\alpha+\xi}; \quad \xi = \{[(\gamma_{12}^{(2)})^p]^2 + [(\gamma_{23}^{(2)})^p]^2\}^{1/2} \tag{39}$$

where μ_0, μ_f and α are material constants. Equations (38) and (39) are sufficient to define, in a unique manner, the components of $[D^{(2)}]$, eqn (2), by following a routine plasticity procedure.

Numerical examples

In order to verify the performance of the proposed framework, an extensive numerical study was undertaken. In particular, a series of in-plane biaxial compression–tension and compression–compression tests was simulated for different orientations of the set of bed joints relative to the loading configuration. The analysis was carried out assuming the brick dimensions as 215 mm × 65 mm and the thickness of the joints as 10 mm. The following material parameters were chosen:

Brick: $E_b = 14{,}700$ MPa, $\nu_b = 0.16$, $f_c = 15.3$ MPa, $f_t = 1.2$ MPa

Mortar: $E_m = 7{,}400$ MPa, $\nu_m = 0.21$,

$\mu_f = 0.73$, $\mu_0 = 0.3\,\mu_f$, $\bar{\mu} = 0.2\mu_f$, $c = 0.78$ MPa, $\alpha = 0.001$.

The material constants describing the response in the elastic range were assumed after Ali and Page (1989). The same reference was used to estimate the values of the last set of parameters, pertaining to the elastoplastic behaviour of mortar. The choice of μ_0, $\bar{\mu}$ and α has been somewhat arbitrary due to the limited experimental information.

Figure 5 shows a set of failure envelopes obtained from the simulation of a series of

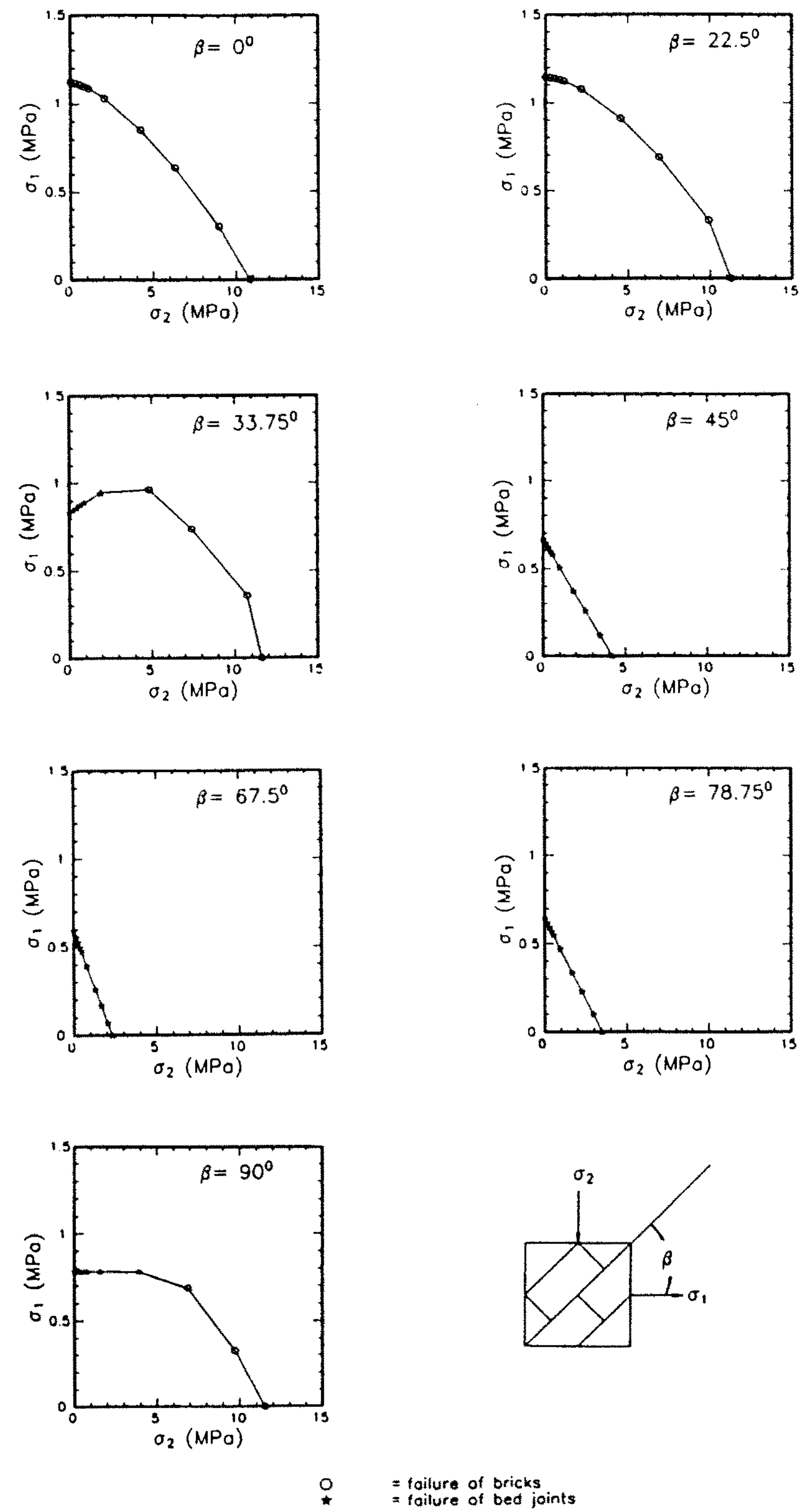

Fig. 5. Failure envelopes for in-plane biaxial compression–tension tests.

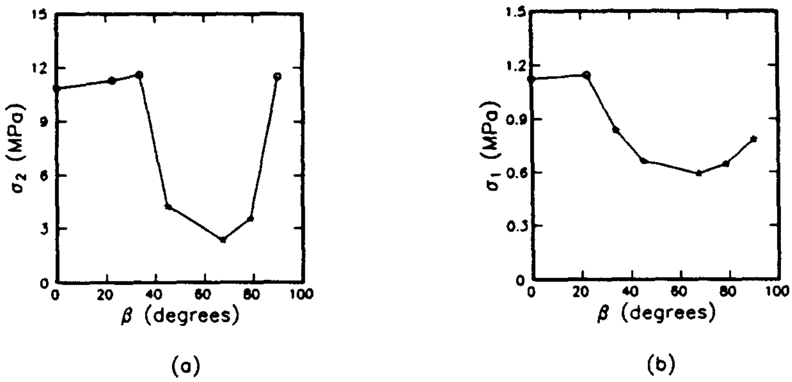

Fig. 6. Variation of uniaxial compressive (a) and tensile (b) strength with the orientation of bed joints.

in-plane biaxial compression–tension tests. The loading process involved a number of trajectories corresponding to a constant compressive–tensile stress ratio (i.e. 0, 0.25, 0.5, 0.75, 1, 2, 5, 10, 30, ∞). The simulations were successively repeated for different orientations of bed joints relative to the direction of the tensile stress (the angle β, ranging from 0° to 90°). Figure 5, apart from defining the set of failure envelopes, provides direct information on the failure mode pertaining to each individual loading history. For low values of β (i.e. $\beta = 0°$ and $\beta = 22.5°$) the collapse of the brickwork is solely induced by the brittle failure of bricks. For $45° < \beta < 78.75°$, the predominant mechanism is the failure of the bed joints, whereas for $\beta = 33.75°$ and $\beta = 90°$ both modes are possible depending on the actual stress ratio.

Figure 6 presents the evolution of **uniaxial** compressive and tensile strength of the brickwork with varying orientations of the bed joints. The results, which are extracted from Fig. 5, indicate that the ultimate strength (both in compression and tension) is the highest for low values of β, i.e. when the failure of masonry is induced by brittle rupture of bricks. As β increases a transition in the failure mode takes place which prompts a drastic reduction in the uniaxial strength. The lowest value corresponds to $\beta \approx 60°$ (bed joints failure).

The results shown in Figs 5 and 6 describe the conditions at failure only, i.e. identify the maximum stress ratio which can be attained for a given stress history. For each loading case, complete stress–strain characteristics are obtained by integration of the constitutive law (19). As an illustration, one such characteristic, corresponding to $\beta = 45°$ and the stress ratio of five, is presented in Fig. 7. A complete deformation history, both on a macroscale and for all the individual constituents, is recorded. Here, the failure of the masonry panel is induced by a ductile failure (shearing) of the bed joints.

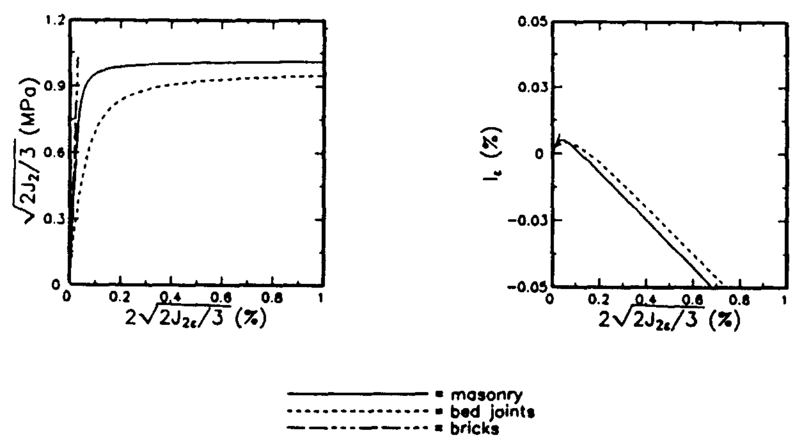

Fig. 7. Material characteristics corresponding to biaxial compression–tension test ($\beta = 45°$; stress ratio 5.0).

The results shown in Fig. 8 correspond to a series of in-plane biaxial compression–compression tests. The loading program was analogous to that for compression–tension and involved a number of stress trajectories at constant vertical to horizontal stress ratio. In this case, the predominant failure mechanism is associated with the brittle failure of the

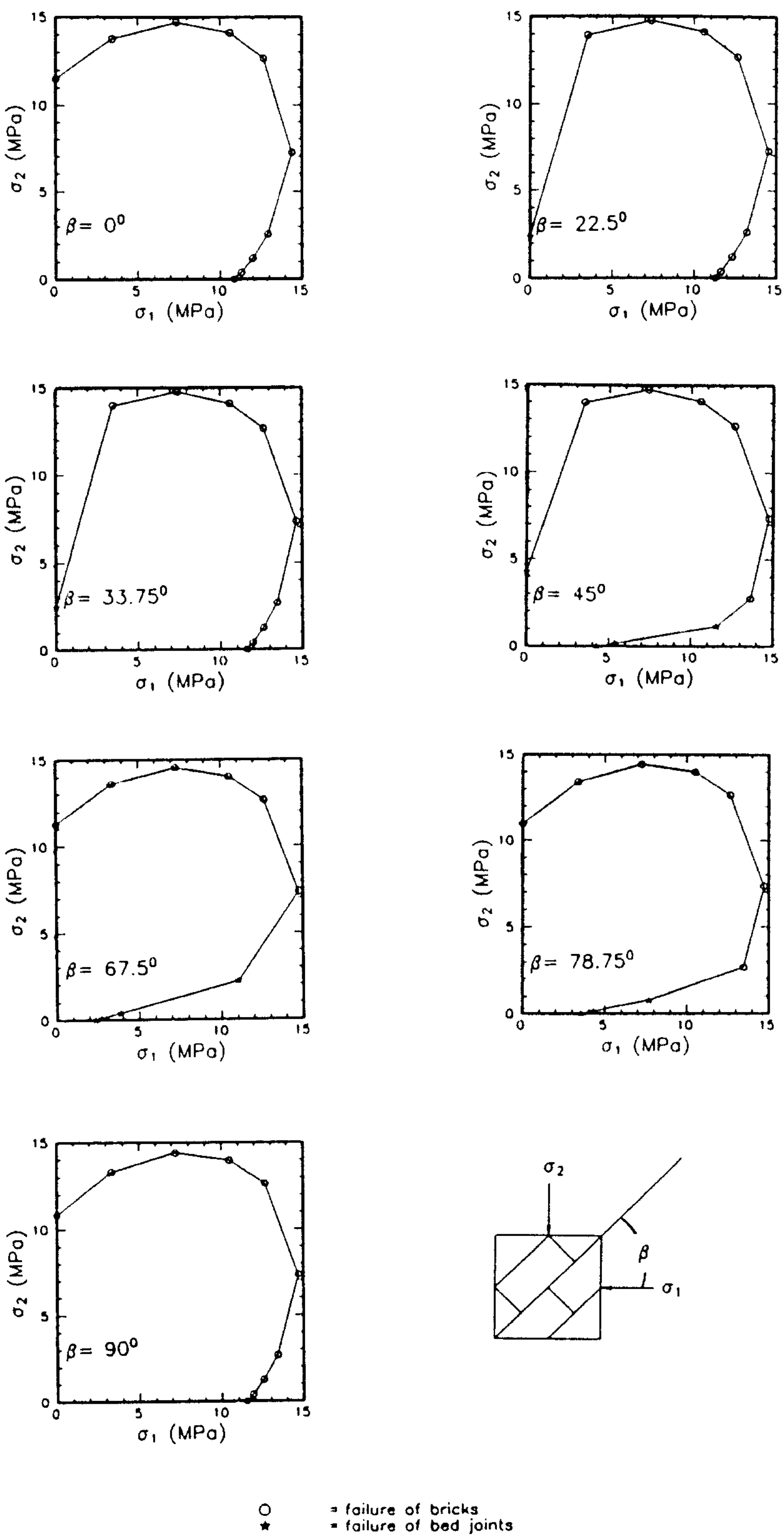

Fig. 8. Failure envelopes for in-plane biaxial compression tests.

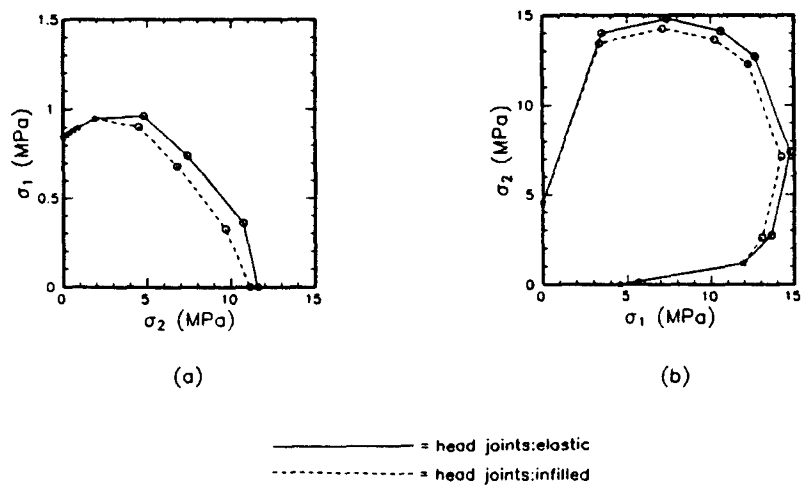

Fig. 9. Influence of head joints on the response under biaxial conditions (a) biaxial compression–tension; (b) biaxial compression.

brick matrix. The failure of bed joints is recorded only for some cases involving extremal values of the stress ratio.

The last aspect of the present analysis is the evaluation of the influence of head joints on the macroscopic failure. The predictions shown in Figs 5–8 have all been based on the assumption that the head joints are linearly elastic. This assumption may not be quite adequate as certain stress trajectories may result in the failure of head joints. Figure 9 shows the predictions for two chosen biaxial tests obtained for the case when the head joints are treated as infilled. Comparing both solutions, i.e. for elastic and infilled head joints, it is evident that the conditions at failure are only marginally affected by the treatment of head joints. In fact, when the failure is initiated in bed joints, the predictions are virtually the same, only when the bricks fail, the predictions show some degree of sensitivity.

Finally, it should be stressed that the numerical analysis presented here is, in fact, of a qualitative nature as no comparison to the experimental data has been provided. The reason is that the experimental reports are usually very fragmentary and there is no comprehensive study giving the adequate information required for quantitative predictions. It should be noted however that the qualitative trends presented here, in the context of both compression–tension and compression–compression tests, are in a close agreement with experimental results reported by Page (1981, 1983).

5. CONCLUSIONS

A mathematical formulation has been presented for describing the average properties of structural masonry. The approach has been derived from the framework of the mechanics of composite media. The proposed constitutive law (19) relates, in a unique manner, the stress rate $\dot{\sigma}$ to strain rate $\dot{\varepsilon}$ averages. Their local counterparts are derived from the corresponding global measures by means of structural matrices whose components are functions of properties of both constituents and their volume contributions. The framework can be incorporated into existing numerical packages to analyze masonry panels of arbitrary geometry. This is feasible providing the characteristic dimension of the elementary volume is much greater than the predominant dimension of the masonry unit.

It has been shown that in the elastic range the brickwork can be considered as an orthotropic medium. The values of elastic constants are strongly influenced by the properties and the thickness of the mortar joints. The failure mechanism consists of a formation of macrocracks in brick matrix or a ductile/brittle failure of the bed joints. The actual failure mode is a function of the imposed loading history. The properties of the head joints have a very limited effect on the macroscopic failure. Thus, for practical purpose, the head joints may be assumed as isotropic linearly elastic.

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